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Multiplicity of (mini-)jets at small x *

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Abstract: We derive closed expressions for the mean and variance of the (mini-)jet multiplicity distribution in hard scattering processes at low x. Here (mini-)jets are defined as those due to initial-state radiation of gluons with transverse momenta greater than some resolution scale μ_{R}, where $\Lambda^2 \ll \mu_{\mathrm{R}}^2 \ll Q^2$, Λ being the intrinsic QCD scale and Q the momentum transfer scale of the hard scattering. Our results are valid to leading order in $\log(1/x)$ but include all sub-leading logarithms of Q^2/μ_{R}^2. As an illustration, we predict the mini-jet multiplicity in Higgs boson production at the Large Hadron Collider.

Keywords: QCD, Deep Inelastic Scattering, Jets, Hadronic Colliders.

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Contents

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1. Introduction

The prediction of jet multiplicities in hard scattering processes represents an important, well-defined challenge to perturbative QCD. Especially challenging is the calculation of the jet multiplicity at small x, that is, in processes where the hard-scattering scale, Q^2, is much less than the overall collision energy-squared s ($x \sim Q^2/s \ll 1$). Such processes include not only the familiar case of deep inelastic lepton scattering at low values of the Bjorken variable, but also Higgs and gauge boson production at high-energy hadron colliders (see e.g. ref. [1]). A large associated jet multiplicity could pose serious triggering and background problems for experiments searching for new physics at the future Large Hadron Collider (LHC).

The particular problem at small x is that higher-order perturbative contributions may become enhanced by factors of $\ln(1/x)$. As a first approximation, one can consider only those terms that are enhanced by a factor of $\ln(1/x)$ for each power of the strong coupling α_{S}. In this approximation, the fully inclusive deep inelastic structure functions satisfy the leading-order Balitskii-Fadin-Kuraev-Lipatov (BFKL) equation [2, 3].

When computing the associated multi-jet production rates at small x [4, 5, 6], one encounters another potentially large logarithmic factor, namely $\ln(Q^2/\mu_{\mathrm{R}}^2)$ where μ_{R} is the minimum transverse momentum that a jet must have in order to be resolved. For mini-jets, by which we mean jets with transverse momenta that are relatively low compared with the hard scattering scale, it clearly becomes important to resum terms enhanced by factors of $\ln(Q^2/\mu_{\rm R}^2)$ as well as $\ln(1/x)$.

In ref. [5], such a resummation was performed in the double-logarithmic (DL) approximation, i.e. keeping only terms of the form $[\alpha_{\rm S}\ln(1/x)\,\ln(Q^2/\mu_{\rm R}^2)]^n$ to all orders. The result was remarkably simple, namely that the mean and variance (mean square fluctuation) of the jet multiplicity are respectively quadratic and cubic functions of $\ln(Q^2/\mu_{\rm R}^2)$, with simple coefficients.

In ref. [6] we extended the results of ref. [5] concerning fixed-multiplicity jet rates to single logarithmic (SL) accuracy, i.e. keeping all terms $[\alpha_{\rm S}\ln(1/x)]^n\,[\ln(Q^2/\mu_{\rm R}^2)]^m$ with $0 < m \leq n$. We also showed that an iterative procedure could be used to obtain the jet multiplicity moments to any desired order in $\alpha_{\rm S}$, to SL accuracy. However, we were not able to derive closed expressions that resum the enhanced terms in the multiplicity moments to all orders. That is done in the present paper.

An alternative approach to hard scattering at small x is the CCFM equation [7, 8] based on angular ordering of gluon emissions. Remarkably, it can be shown [9] that for sufficiently inclusive observables the CCFM formalism leads to the same results as the BFKL equation. The jet multiplicities calculated in the present paper fall into this class of observables. The BFKL formalism turns out to be technically simpler and therefore we adopt it here.

The paper is organized as follows. First, in sect. 2, we solve the equation derived in ref. [6] for the jet-rate generating function, by expressing the solution as an inverse Laplace transform. In sect. 3 we obtain the jet multiplicity moments by saddle-point evaluation of the inverse Laplace transform. The results are discussed in sect. 4. We find that the simple quadratic and cubic forms of the mean and variance found in the DL approximation remain valid to SL precision. The coefficients, while not particularly simple, can be expressed straightforwardly in terms of the Lipatov anomalous dimension [2] and its derivatives.

As an illustrative application of our results, in sect. 5 we calculate the mean and variance of the mini-jet multiplicity in Higgs boson production at the LHC, as functions of the Higgs mass.

2. Jet-rate generating function

We write the contribution to the gluon structure function at scale Q^2, $F(x,Q^2)$, in which r final-state mini-jets are resolved with transverse momenta greater than $\mu_{\rm R}$, in the form

$$F^{(r\,{\rm jet})}(x,Q^2,\mu_{\rm R}^2) = F(x,\mu_{\rm R}^2)\otimes G^{(r)}(x,T) \equiv \int_x^1 \frac{dz}{z} F(z,\mu_{\rm R}^2)G^{(r)}\left(\frac{x}{z},T\right), \qquad (2.1)$$

where the coefficient $G^{(r)}$ is a function of x and $T = \ln(Q^2/\mu_{\rm R}^2)$ to be determined .

The Mellin transformation

$$f_\omega(\ldots) = \int_0^1 dx\, x^\omega f(x,\ldots)\,, \tag{2.2}$$

with inverse

$$f(x,\ldots) = \frac{1}{2\pi i}\int_C d\omega\, x^{-\omega-1} f_\omega(\ldots)\,, \tag{2.3}$$

where the contour C is parallel to the imaginary axis and to the right of all singularities of the integrand, converts the convolution in eq. (2.1) into a simple product:

$$F_\omega^{(r\,\mathrm{jet})}(Q^2,\mu_{\mathrm{R}}^2) = F_\omega(\mu_{\mathrm{R}}^2)\, G_\omega^{(r)}(T)\,. \tag{2.4}$$

Furthermore the evolution of the structure function in ω-space is simply given by

$$F_\omega(Q^2) = \exp\left[\gamma_{\mathrm{L}}(\bar\alpha_{\mathrm{S}}/\omega)\, T\right] F_\omega(\mu_{\mathrm{R}}^2)\,, \tag{2.5}$$

where γ_{L} is the Lipatov anomalous dimension:

$$\gamma_{\mathrm{L}}(\bar\alpha_{\mathrm{S}}/\omega) = \frac{\bar\alpha_{\mathrm{S}}}{\omega} + 2\zeta(3)\left(\frac{\bar\alpha_{\mathrm{S}}}{\omega}\right)^4 + 2\zeta(5)\left(\frac{\bar\alpha_{\mathrm{S}}}{\omega}\right)^6 + 12[\zeta(3)]^2\left(\frac{\bar\alpha_{\mathrm{S}}}{\omega}\right)^7 + \cdots\,, \tag{2.6}$$

which is the solution of the equation

$$\omega = -\bar\alpha_{\mathrm{S}}\left[2\gamma_{\mathrm{E}} + \psi(\gamma) + \psi(1-\gamma)\right] \equiv \bar\alpha_{\mathrm{S}}\,\chi(\gamma)\,, \tag{2.7}$$

where $\bar\alpha_{\mathrm{S}} = 3\alpha_{\mathrm{S}}/\pi$, ψ being the digamma function and $\gamma_{\mathrm{E}} = -\psi(1)$ the Euler constant.

Introducing the jet-rate generating function

$$G_\omega(u,T) = \sum_{r=0}^{\infty} u^r G_\omega^{(r)}(T)\,, \tag{2.8}$$

we can write the r-jet rate, i. e. the fraction of events with r mini-jets, as

$$R_\omega^{(r\,\mathrm{jet})}(Q^2,\mu_{\mathrm{R}}^2) = \frac{F_\omega^{(r\,\mathrm{jet})}(Q^2,\mu_{\mathrm{R}}^2)}{F_\omega(Q^2)} = \frac{1}{r!}\,\frac{\partial^r}{\partial u^r} R_\omega(u,T)\bigg|_{u=0}\,, \tag{2.9}$$

where the first equation defines $R_\omega^{(r\,\mathrm{jet})}(Q^2,\mu_{\mathrm{R}}^2)$ and

$$R_\omega(u,T) = \exp\left[-\gamma_{\mathrm{L}}(\bar\alpha_{\mathrm{S}}/\omega)\, T\right]\, G_\omega(u,T)\,. \tag{2.10}$$

We showed in ref. [6] that $G_\omega(u,T)$ is given by

$$G_\omega(u,T) = 1 + \int_0^T dt\, g_\omega(u,t,T)\,, \tag{2.11}$$

where the unintegrated function $g_\omega(u,t,T)$ satisfies the integro-differential equation

$$g_\omega = \frac{u\bar\alpha_{\mathrm{S}}}{\omega+\bar\alpha_{\mathrm{S}} t}\left[1 + \int_0^T dt'\, g_\omega\left(u, \max\{t,t'\}, T\right) + 2\sum_{m=1}^{\infty}\zeta(2m+1)\,\frac{\partial^{2m} g_\omega}{\partial t^{2m}}\right]. \tag{2.12}$$

Rearranging terms and differentiating with respect to t, this gives

$$\frac{\partial}{\partial t}\left\{\left[1-(u-1)t\frac{\bar{\alpha}_{\rm S}}{\omega}\right]g_\omega\right\} = -u\frac{\bar{\alpha}_{\rm S}}{\omega}\left[1-2\sum_{m=1}^{\infty}\zeta(2m+1)\,\frac{\partial^{2m+1}}{\partial t^{2m+1}}\right]g_\omega\,. \tag{2.13}$$

To find g_ω we first recall the solution in double-logarithmic approximation [5],

$$g_\omega^{\rm DL}(u,t,T) = u\frac{\bar{\alpha}_{\rm S}}{\omega}\left[1-(u-1)t\frac{\bar{\alpha}_{\rm S}}{\omega}\right]^{\frac{1}{u-1}}\left[1-(u-1)T\frac{\bar{\alpha}_{\rm S}}{\omega}\right]^{\frac{-u}{u-1}}\,. \tag{2.14}$$

Notice that this expression vanishes, together with all its t-derivatives, at the point $t=t_0=\omega/[\bar{\alpha}_{\rm S}(u-1)]$ as $u\to 1^+$. We shall assume this is also true to SL accuracy, for some finite value of t_0, and then verify that the solution is consistent with perturbation theory. The equation (2.13) can now be solved by introducing a Laplace transformation of the form

$$\tilde{g}_\omega(u,\gamma,T) = \int_{-\infty}^{t_0} dt\, g_\omega(u,t,T)\,e^{\gamma t}\,, \tag{2.15}$$

with inverse

$$g_\omega(u,t,T) = \frac{1}{2\pi i}\int_\Gamma d\gamma\,\tilde{g}_\omega(u,\gamma,T)\,e^{-\gamma t}\,, \tag{2.16}$$

where the contour Γ is parallel to the imaginary axis and to the left of all singularities of the integrand. This gives

$$\tilde{g}_\omega(u,\gamma,T) = \tilde{g}_\omega(u,\gamma_0,T)\,e^{\phi_\omega(u,\gamma)}\,, \tag{2.17}$$

where

$$\begin{aligned}\phi_\omega(u,\gamma) &= \frac{u}{u-1}\int_{\gamma_0}^{\gamma} d\gamma'\,\left[\frac{\omega}{\bar{\alpha}_{\rm S}u}-\chi(\gamma')\right] \\ &= \frac{u}{u-1}\left[\left(\frac{\omega}{\bar{\alpha}_{\rm S}u}+2\gamma_{\rm E}\right)(\gamma-\gamma_0)-\ln\frac{\Gamma(1-\gamma)\Gamma(\gamma_0)}{\Gamma(\gamma)\Gamma(1-\gamma_0)}\right]\,,\end{aligned} \tag{2.18}$$

χ being the Lipatov characteristic function, see eq. (2.7). To determine $\tilde{g}_\omega(u,\gamma_0,T)$ we make use of the boundary condition provided by eq. (2.12),

$$g_\omega(u,T,T) = \frac{u\bar{\alpha}_{\rm S}}{\omega+\bar{\alpha}_{\rm S}T}\left[1+g_\omega t+2\sum_{m=1}^{\infty}\zeta(2m+1)\,\frac{\partial^{2m}g_\omega}{\partial t^{2m}}\right]_{t=T}\,, \tag{2.19}$$

which tells us that

$$\frac{1}{2\pi i}\tilde{g}_\omega(u,\gamma_0,T)\int_\Gamma d\gamma\left[\frac{\omega}{\bar{\alpha}_{\rm S}u}-\frac{u-1}{u}T+\frac{1}{\gamma}-\chi(\gamma)\right]\,e^{-\gamma T+\phi_\omega(u,\gamma)} = 1\,. \tag{2.20}$$

Deleting a total derivative from the integrand, since endpoint contributions must vanish, we obtain simply

$$\frac{1}{2\pi i}\tilde{g}_\omega(u,\gamma_0,T) = \left[I_\omega(u,T)\right]^{-1}\,, \tag{2.21}$$

where

$$I_\omega(u,T) = \int_\Gamma \frac{d\gamma}{\gamma}\, e^{-\gamma T + \phi_\omega(u,\gamma)}\,. \tag{2.22}$$

Hence

$$g_\omega(u,t,T) = \frac{1}{I_\omega(u,T)} \int_\Gamma d\gamma\, e^{-\gamma t + \phi_\omega(u,\gamma)} \tag{2.23}$$

so that finally

$$G_\omega(u,T) = 1 + \int_0^T dt\, g_\omega(u,t,T) = \frac{I_\omega(u,0)}{I_\omega(u,T)}\,. \tag{2.24}$$

3. Jet multiplicity moments

The moments of the jet multiplicity distribution are obtained by successive differentiation of the generating function at $u = 1$:

$$\overline{r(r-1)\ldots(r-s+1)}_\omega = \left.\frac{\partial^s R_\omega}{\partial u^s}\right|_{u=1} = \exp\left[-\gamma_{\rm L}(\bar\alpha_{\rm S}/\omega)\,T\right] \left.\frac{\partial^s G_\omega}{\partial u^s}\right|_{u=1}\,. \tag{3.1}$$

To compute these quantities to all orders from the expression (2.24) we can use the method of steepest descent. It is clear from eq. (2.18) that the saddle point of $\phi_\omega(u,\gamma)$ occurs at $\gamma = \gamma_s$ where $\omega = \bar\alpha_{\rm S} u \chi(\gamma_s)$, or in other words, from eq. (2.7), $\gamma_s = \gamma_{\rm L}(\bar\alpha_{\rm S} u/\omega)$. Choosing for convenience $\gamma_0 = \gamma_s$, we have

$$\phi_\omega(u,\gamma) = -\frac{u}{u-1}\sum_{n=2}^{\infty}\frac{1}{n!}(\gamma-\gamma_s)^n \chi^{(n-1)}(\gamma_s)\,. \tag{3.2}$$

Now we have to evaluate integrals of the form[1]

$$\begin{aligned}
\int_\Gamma d\gamma\, f(\gamma)\, e^{\phi_\omega(u,\gamma)} &= \int_\Gamma d\gamma \sum_{m=0}^{\infty}\frac{f_s^{(m)}}{m!}(\gamma-\gamma_s)^m\, e^{\phi_\omega(u,\gamma)} \\
&= \left\{ f_s + \frac{(u-1)}{u\chi'_s}\left(\frac{1}{2}f''_s - \frac{1}{2}\frac{\chi''_s}{\chi'_s}f'_s\right) + \right. \\
&\quad + \left(\frac{u-1}{u\chi'_s}\right)^2 \left[\frac{1}{8}f''''_s - \frac{5}{12}\frac{\chi''_s}{\chi'_s}f'''_s - \frac{1}{4}\frac{\chi'''_s}{\chi'_s}f''_s + \frac{5}{8}\frac{\chi''^2_s}{\chi'^2_s}f''_s - \right. \\
&\quad \left.\left. - \frac{1}{8}\frac{\chi''''_s}{\chi'_s}f'_s + \frac{2}{3}\frac{\chi''_s\chi'''_s}{\chi'^2_s}f'_s - \frac{5}{8}\left(\frac{\chi''_s}{\chi'_s}\right)^3 f'_s\right] + \cdots\right\} \int_\Gamma d\gamma\, e^{\phi_\omega(u,\gamma)}
\end{aligned} \tag{3.3}$$

where $f_s = f(\gamma_s)$, $\chi'_s = \chi'(\gamma_s)$, etc., and the dots represent terms of higher order in $(u-1)$. To express everything in terms of $f(\gamma_{\rm L})$, $\chi'(\gamma_{\rm L})$, etc., where $\gamma_{\rm L} \equiv \gamma_{\rm L}(\bar\alpha_{\rm S}/\omega)$, we use the results in the appendix to write

$$\begin{aligned}
\gamma_s &= \gamma_{\rm L} + \gamma'_{\rm L}\,(u-1)\frac{\bar\alpha_{\rm S}}{\omega} + \frac{1}{2}\gamma''_{\rm L}\,(u-1)^2\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^2 + \cdots \\
&= \gamma_{\rm L} - (u-1)\frac{\chi}{\chi'} + (u-1)^2\left(\frac{\chi}{\chi'} - \frac{\chi^2\chi''}{2\chi'^3}\right) + \cdots
\end{aligned} \tag{3.4}$$

[1]The saddle-point method has been applied to similar integrals in ref. [10].

where now $\chi = \chi(\gamma_{\rm L})$, etc., and we can use this to expand eq. (3.3).

In the present case we have

$$f(\gamma) = \frac{1}{\gamma} e^{-\gamma T}\,. \tag{3.5}$$

Hence from eqs. (2.24), (3.1) and (3.3) we obtain for the mean number of jets,

$$\overline{r}_\omega = -\frac{1}{\chi'}\left(\frac{1}{\gamma_{\rm L}} + \frac{\chi''}{2\chi'} + \chi\right) T - \frac{1}{2\chi'} T^2 \tag{3.6}$$

and for the variance $\sigma_\omega^2 \equiv \overline{r^2}_\omega - \overline{r}_\omega^2$,

$$\begin{aligned}\sigma_\omega^2 = \frac{1}{\chi'}\Bigg[&\frac{1}{\gamma_{\rm L}} + \chi - \frac{4}{\gamma_{\rm L}^3\chi'} - \frac{2\chi}{\gamma_{\rm L}^2\chi'} - \frac{\chi^2\chi''}{\chi'^2} - \frac{3\chi''}{\gamma_{\rm L}^2\chi'^2} - \frac{2\chi\chi''}{\gamma_{\rm L}\chi'^2} + \frac{\chi''}{2\chi'} - \\ &- \frac{2\chi\chi''^2}{\chi'^3} - \frac{2\chi''^2}{\gamma_{\rm L}\chi'^3} - \frac{5\chi''^3}{4\chi'^4} + \frac{\chi\chi'''}{\chi'^2} + \frac{\chi'''}{\gamma_{\rm L}\chi'^2} + \frac{4\chi''\chi'''}{3\chi'^3} - \frac{\chi''''}{4\chi'^2}\Bigg] T + \\ &+ \frac{1}{\chi'}\left(\frac{1}{2} - \frac{1}{\gamma_{\rm L}^2\chi'} - \frac{\chi\chi''}{\chi'^2} - \frac{\chi''}{\gamma_{\rm L}\chi'^2} - \frac{\chi''^2}{\chi'^3} + \frac{\chi'''}{2\chi'^2}\right) T^2 - \frac{\chi''}{3\chi'^3} T^3\,. \end{aligned} \tag{3.7}$$

Eqs. (3.6) and (3.7) are convenient for numerical calculation, since they only involve the evaluation of polygamma functions. To compare with perturbative expansions, we can use the results in the appendix to rewrite them in terms of $\gamma_{\rm L}$ and its derivatives:

$$\overline{r}_\omega = \left[-1 + \gamma'_{\rm L} + \frac{\bar\alpha_{\rm S}}{\omega}\left(\frac{\gamma'_{\rm L}}{\gamma_{\rm L}} - \frac{\gamma''_{\rm L}}{2\gamma'_{\rm L}}\right)\right]\frac{\bar\alpha_{\rm S}}{\omega} T + \frac{1}{2}\gamma'_{\rm L}\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^2 T^2\,, \tag{3.8}$$

$$\begin{aligned}\sigma_\omega^2 = \Bigg[&-1 + \gamma'_{\rm L} + \frac{\bar\alpha_{\rm S}}{\omega}\left(-\frac{2}{\gamma_{\rm L}} + \frac{3\gamma'_{\rm L}}{\gamma_{\rm L}} + \gamma''_{\rm L} - \frac{3\gamma''_{\rm L}}{2\gamma'_{\rm L}}\right) + \\ &+ \left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^2\left(\frac{6\gamma'_{\rm L}}{\gamma_{\rm L}^2} - \frac{2\gamma'^2_{\rm L}}{\gamma_{\rm L}^2} + \frac{2\gamma''_{\rm L}}{\gamma_{\rm L}} - \frac{2\gamma''_{\rm L}}{\gamma_{\rm L}\gamma'_{\rm L}} - \frac{\gamma''^2_{\rm L}}{\gamma'^3_{\rm L}} + \frac{\gamma''^2_{\rm L}}{\gamma'^2_{\rm L}} + \frac{2\gamma'''_{\rm L}}{3\gamma'^2_{\rm L}} - \frac{\gamma'''_{\rm L}}{\gamma'_{\rm L}}\right) + \\ &+ \left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^3\left(\frac{-4\gamma'^2_{\rm L}}{\gamma_{\rm L}^3} + \frac{3\gamma''_{\rm L}}{\gamma_{\rm L}^2} + \frac{\gamma''^2_{\rm L}}{\gamma_{\rm L}\gamma'^2_{\rm L}} + \frac{\gamma''^3_{\rm L}}{\gamma'^4_{\rm L}} - \frac{\gamma'''_{\rm L}}{\gamma_{\rm L}\gamma'_{\rm L}} - \frac{7\gamma''_{\rm L}\gamma'''_{\rm L}}{6\gamma'^3_{\rm L}} + \frac{\gamma''''_{\rm L}}{4\gamma'^2_{\rm L}}\right)\Bigg]\frac{\bar\alpha_{\rm S}}{\omega} T + \\ &+ \left[-1 + \frac{3\gamma'_{\rm L}}{2} + \frac{\bar\alpha_{\rm S}}{\omega}\left(\gamma''_{\rm L} + \frac{2\gamma'_{\rm L}}{\gamma_{\rm L}} - \frac{\gamma''_{\rm L}}{\gamma'_{\rm L}}\right) + \left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^2\left(-\frac{\gamma'^2_{\rm L}}{\gamma_{\rm L}^2} + \frac{\gamma''_{\rm L}}{\gamma_{\rm L}} + \frac{\gamma''^2_{\rm L}}{2\gamma'^2_{\rm L}} - \frac{\gamma'''_{\rm L}}{2\gamma'_{\rm L}}\right)\right] \times \\ &\times \left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^2 T^2 + \frac{1}{3}\left(2\gamma'_{\rm L} + \frac{\bar\alpha_{\rm S}}{\omega}\gamma''_{\rm L}\right)\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^3 T^3\,. \end{aligned} \tag{3.9}$$

Using the perturbative expansion (2.6) for $\gamma_{\rm L}$, we find

$$\begin{aligned}\overline{r}_\omega = &\frac{\bar\alpha_{\rm S}}{\omega} T + \frac{1}{2}\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^2 T^2 + 2\zeta(3)\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^4 T + \\ &+ 4\zeta(3)\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^5 T^2 - 8\zeta(5)\left(\frac{\bar\alpha_{\rm S}}{\omega}\right)^6 T + \cdots \end{aligned} \tag{3.10}$$

and

$$\sigma_\omega^2 = \frac{\bar{\alpha}_{\rm S}}{\omega}T + \frac{3}{2}\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^2 T^2 + \frac{2}{3}\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^3 T^3 - 2\zeta(3)\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^4 T + \\ + 12\zeta(3)\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^5 T^2 - \left(8\zeta(5) - \frac{40}{3}\zeta(3)T^2\right)\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^6 T + \cdots\,, \tag{3.11}$$

which do indeed agree with the expansions obtained in ref. [6].

4. Discussion of the jet multiplicity moments

We see from eqs. (3.6) and (3.7) that it remains true to all orders to SL precision that, as in the DL approximation [5], the mean number of jets is a quadratic function of T while the variance is a cubic function of T. Thus the distribution of jet multiplicity at small x and large T remains narrow, in the sense that its r.m.s. width increases less rapidly than its mean with increasing T.

It is interesting to see how rapidly the perturbative expansions for the mean number of mini-jets and its variance converge. We first introduce the notation

$$\overline{r}_\omega = a_1\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)T + a_2\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^2 T^2 \tag{4.1}$$

and

$$\sigma_\omega^2 = b_1\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)T + b_2\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^2 T^2 + b_3\left(\frac{\bar{\alpha}_{\rm S}}{\omega}\right)^3 T^3\,. \tag{4.2}$$

Fig. 1 shows the coefficients a_i $(i = 1, 2)$ as functions of $\bar{\alpha}_{\rm S}/\omega$, including the exact dependence according to eq. (3.6) as well as the perturbative expressions (3.8) expanded to different orders. The coefficients b_i $(i = 1, 2, 3)$ are shown in fig. 2.

The coefficients in the mean jet multiplicity $\overline{r}_\omega$ are extremely well approximated by the perturbative expressions for small values of $\bar{\alpha}_{\rm S}/\omega$, up to 0.3. In this region, the coefficients remain close to the (constant) values predicted by the double-logarithmic approximation. Above $\bar{\alpha}_{\rm S}/\omega \simeq 0.3$, where the coefficients start to diverge, the perturbative expansion converges rather slowly. A similar behaviour is observed for the coefficients in the variance σ_ω^2. The perturbative expansion converges very rapidly for small $\bar{\alpha}_{\rm S}/\omega$, in this case up to ~ 0.25. Above $\bar{\alpha}_{\rm S}/\omega \simeq 0.3$, the convergence is again rather poor.

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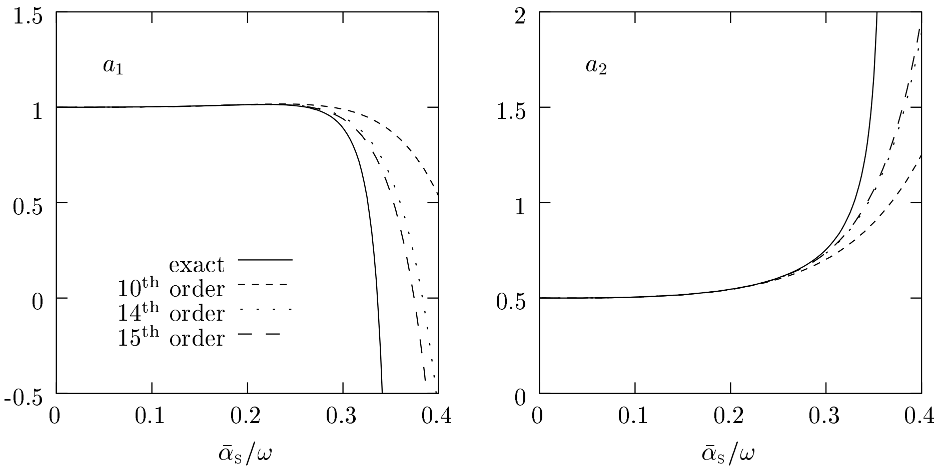

Figure 1: Coefficients in the mean jet multiplicity $\overline{r}_\omega$ as functions of $\bar{\alpha}_{\rm S}/\omega$, as defined in (4.1). The solid line represents the exact coefficients as calculated from (3.6), the other lines are obtained by expanding (3.8) to 10th, 14th, and 15th order in $\bar{\alpha}_{\rm S}/\omega$.

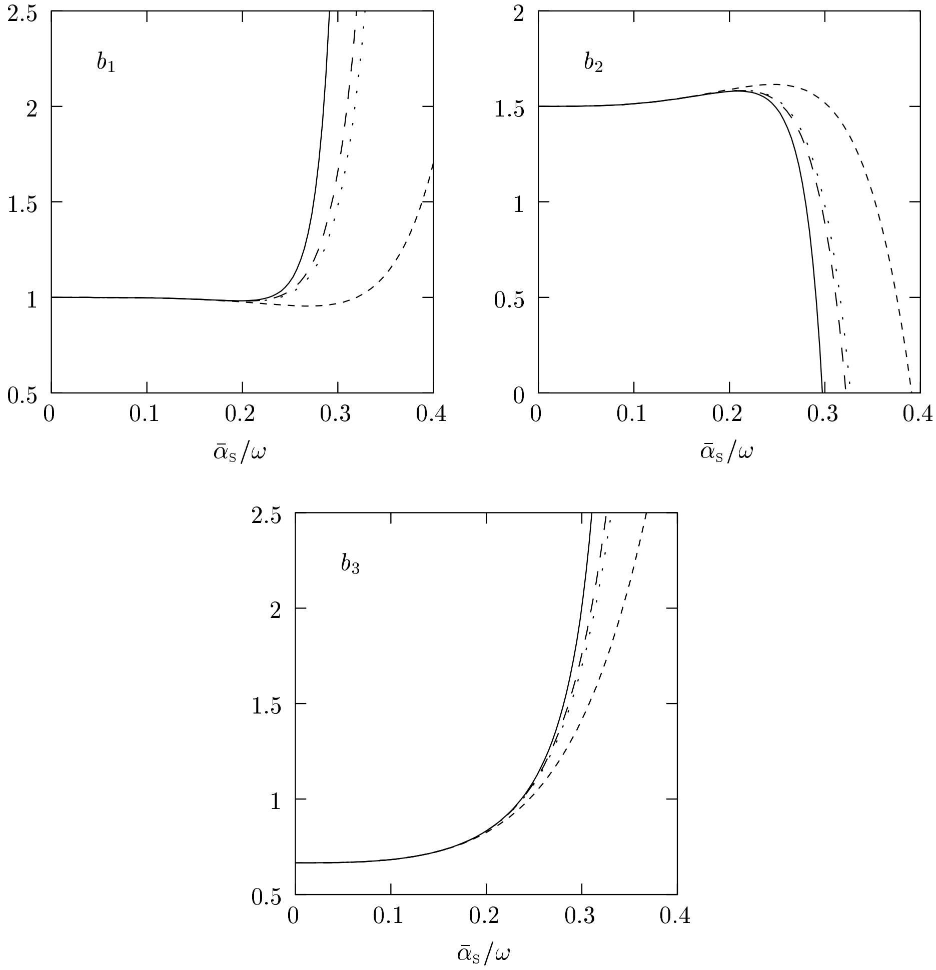

Figure 2: Coefficients in the variance σ^2_ω as functions of $\bar{\alpha}_{\rm S}/\omega$, as defined in (4.2). Lines as in fig. 1.

The singularities in the coefficients arise from the fact that eq. (2.7) does not have real solutions for $\bar{\alpha}_S/\omega > (4\ln 2)^{-1} = 0.361$. This bound will be raised [10] when NLO corrections to the BFKL equation [11, 12] are included. We therefore expect higher order corrections to the BFKL equation to have a significant effect on the jet multiplicity moments for values of $\bar{\alpha}_S/\omega$ above ~ 0.3.

For practical applications, we need to compute the multiplicity moments as functions of x. This is done most conveniently using the perturbative expansions (3.10) and (3.11), extended to sufficiently high order, and performing the inverse Mellin transformation term by term using

$$\frac{1}{2\pi i}\int_C d\omega\, x^{-\omega-1}\left(\frac{\bar{\alpha}_S}{\omega}\right)^n = \frac{\bar{\alpha}_S}{x}\frac{[\bar{\alpha}_S\ln(1/x)]^{n-1}}{(n-1)!}\,. \tag{4.3}$$

The factorial suppression of high orders in x-space removes the convergence problems that were encountered in ω-space.

To illustrate the behaviour in x-space, we consider the mini-jet multiplicity associated with pointlike scattering on the gluonic component of the proton at small x. The mean number of mini-jets considered as a function of x is given by

$$n(x) = \frac{F(x,Q^2)\otimes\overline{r}(x)}{F(x,Q^2)}\,. \tag{4.4}$$

where $\overline{r}(x)$ is the inverse Mellin transform of $\overline{r}_\omega$, obtained by means of eq. (4.3). Similarly the dispersion σ_n in the mean number of mini-jets as a function of x is

$$\sigma_n^2(x) = \frac{F(x,Q^2)\otimes\overline{r^2}(x)}{F(x,Q^2)} - [n(x)]^2\,. \tag{4.5}$$

To perform the convolution in eqs. (4.4), (4.5) we used the leading-order MRST gluon distribution [13]. Fig. 3 shows the resulting x-dependence of the mean number of mini-jets $n(x)$ and its dispersion $\sigma_n(x)$ for $Q=100\,\mathrm{GeV}$ and two different values of the resolution scale μ_R. We see that the results remain close to the predictions of the double-logarithmic approximation at all but the very smallest values of x and μ_R. The convergence of the perturbation series in x-space is so good that there is no visible difference between the results at 5th and 15th order for the ranges of x and μ_R shown.

We should emphasise that the results in fig. 3 are not directly comparable with data on deep inelastic lepton scattering (DIS). To make quantitative predictions for DIS one would need to perform a further convolution with an impact factor that represents the "quark box" coupling of the gluon to the virtual photon. One should also take into account the production of mini-jets from the quark box, together with non-perturbative effects such as mini-jet emission from the proton remnant.

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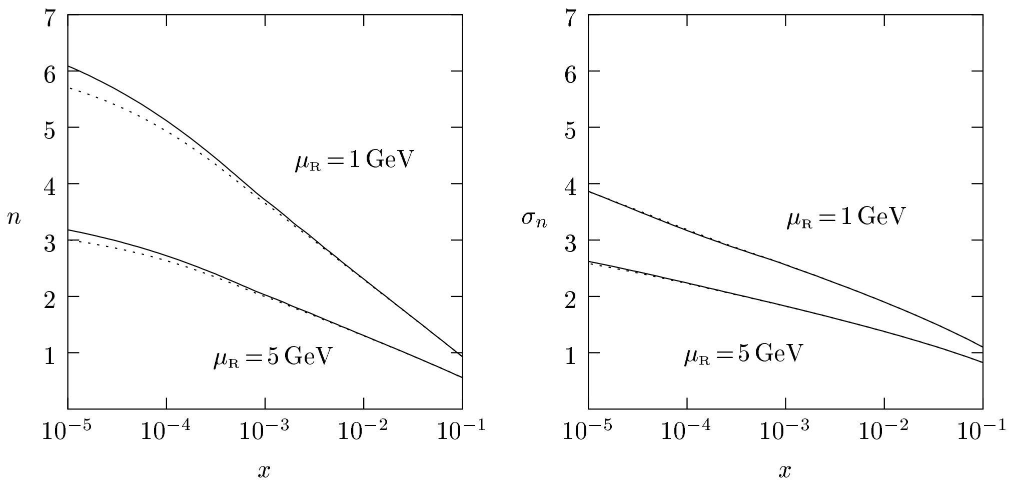

Figure 3: The mean and dispersion of the mini-jet multiplicity as functions of x at $Q = 100\,\mathrm{GeV}$ for two different resolution scales μ_R. The solid lines show the SL results including the 15th order in perturbation theory, the dashed lines correspond to the DL approximation.

5. Mini-jet multiplicity in Higgs production at LHC

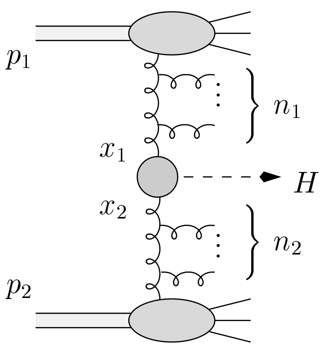

Figure 4: Higgs boson production by gluon-gluon fusion.

The dominant production process for a standard model Higgs boson at the LHC is expected to be gluon-gluon fusion. This process is illustrated in fig. 4. The production cross section for a Higgs boson of mass M and rapidity y by gluon-gluon fusion in proton-proton collisions at centre-of mass energy $\sqrt{s}$ takes the form [1]

$$\frac{d\sigma}{dy} = F(x_1, M^2)\, F(x_2, M^2)\, C(M^2)\,, \tag{5.1}$$

where

$$x_1 = \frac{M}{\sqrt{s}}\, e^{y}\,, \qquad\qquad x_2 = \frac{M}{\sqrt{s}}\, e^{-y}\,, \tag{5.2}$$

and for LHC $\sqrt{s} = 14\,\mathrm{TeV}$. C represents the $gg \to H$ vertex, which is perturbatively calculable as an intermediate top-quark loop. As in DIS, a more careful treatment would involve replacing the Higgs production vertex $C(M^2)$ by an impact factor $C(M^2, k_1^2, k_2^2)$ and convoluting it with unintegrated gluon densities taken at the off-shell gluon virtualities k_1^2 and k_2^2, respectively. The dependence of the impact factor $C(M^2, k_1^2, k_2^2)$ on these virtualities is expected to be weak, and we have neglected it to arrive at eq. (5.1). Then C cancels in the mean number of mini-jets and its dispersion, and we do not need to know its detailed form. In this approximation, we can compute the associated mini-jet multiplicity in Higgs production quite simply using the machinery developed in the previous section.

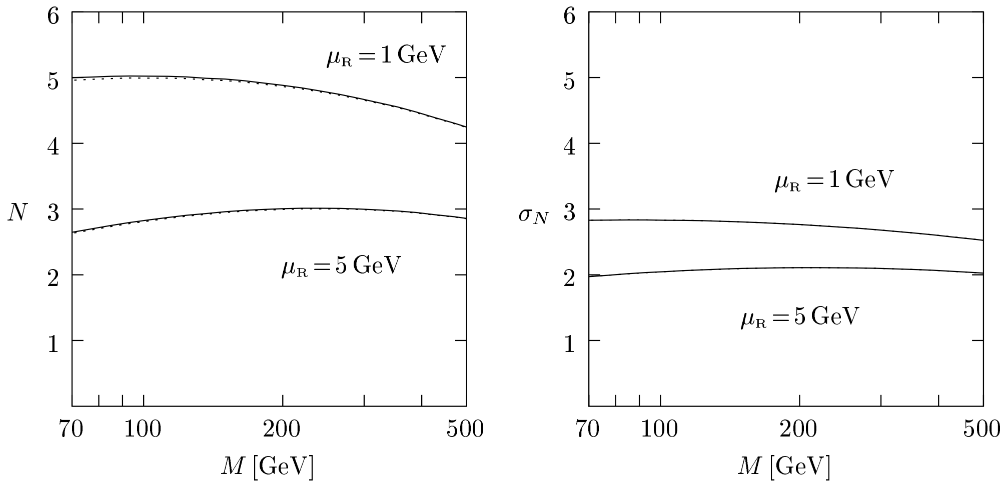

Figure 5: The mean value and dispersion of the number of (mini-)jets in central Higgs production at LHC for two different resolution scales μ_R. Solid lines show the SL results up to the 15th order in perturbation theory, dashed lines correspond to the DL approximation.

We will consider central production of the Higgs boson ($y = 0$), and therefore we have $x_1 = x_2 = x$ where $x = M/\sqrt{s}$. Since the gluon emissions in the upper and lower parts of fig. 4 are independent, we can simply add the numbers $n_1 = n(x_1)$ and $n_2 = n(x_2)$ of mini-jets, and the mean multiplicity N of associated mini-jets becomes[2]

$$N(x) = n_1 + n_2 = 2n(x)\,, \tag{5.3}$$

where $n(x)$ can be calculated as in (4.4) after replacing Q^2 by M^2. Similarly, the variance is

$$\sigma_N^2(x) = \sigma_n^2(x_1) + \sigma_n^2(x_2) = 2\sigma_n^2(x)\,. \tag{5.4}$$

Again, σ_n^2 can be obtained as in eq. (4.5).

We have calculated the dependence of N and σ_N on the Higgs mass M and our numerical results are shown in fig. 5. Here, the DL results give an excellent approximation and the SL terms are less significant. We see that the mini-jet multiplicity and its dispersion are rather insensitive to the Higgs mass at the energy of the LHC.

6. Conclusions

We have derived expressions for the mean and variance of the jet multiplicity distribution at small x, including resummation of leading logarithms of x and all (leading and sub-leading) logarithms of Q^2/μ_R^2. Our results have been derived using the BFKL equation, but are expected to hold in the CCFM formalism as well.

Considered as functions of ω, the moment variable Mellin-conjugate to x, our expressions exhibit bad behaviour at large values of $\bar{\alpha}_S/\omega$, which is associated with the

[2]Again we do not count any jets emerging from the proton remnants.

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singularity of the leading-order Lipatov anomalous dimension γ_{L} at $\bar{\alpha}_{\mathrm{S}}/\omega = (4\ln 2)^{-1}$. We would expect this behaviour to be modified strongly by higher-order corrections. Although the next-to-leading corrections to γ_{L} are known, a full calculation of the corresponding corrections to the associated jet multiplicity has not been performed and would appear much more difficult.

In x-space we have only been able to find the jet multiplicity and its variance as perturbative expressions. These are obtained after expanding the closed expressions in ω-space to sufficiently high order and applying the inverse Mellin transformation term by term. For all practical applications this is sufficient since the perturbative series in x-space turns out to be very rapidly convergent.

The multiplicity of mini-jets at small x is an observable that can also be computed in Monte Carlo simulations of BFKL dynamics [14]. It would be interesting to compare the corresponding results with our analytic results.

A. Lipatov anomalous dimension and characteristic function

For changing from γ_{L} and its derivatives to χ and its derivatives we need the following relations, which are easily obtained by repeated differentiation of the first:

$$\chi(\gamma_{\mathrm{L}}) = \frac{\omega}{\bar{\alpha}_{\mathrm{S}}} \tag{A.1}$$

$$\chi'(\gamma_{\mathrm{L}}) = -\left(\frac{\omega}{\bar{\alpha}_{\mathrm{S}}}\right)^2 \frac{1}{\gamma_{\mathrm{L}}'} \tag{A.2}$$

$$\chi''(\gamma_{\mathrm{L}}) = \left(\frac{\omega}{\bar{\alpha}_{\mathrm{S}}}\right)^3 \frac{1}{\gamma_{\mathrm{L}}'^3} \left(2\gamma_{\mathrm{L}}' + \frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\gamma_{\mathrm{L}}''\right) \tag{A.3}$$

$$\chi'''(\gamma_{\mathrm{L}}) = \left(\frac{\omega}{\bar{\alpha}_{\mathrm{S}}}\right)^4 \frac{1}{\gamma_{\mathrm{L}}'^5} \left[-6\gamma_{\mathrm{L}}'^2 - 3\gamma_{\mathrm{L}}''^2 - 6\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\gamma_{\mathrm{L}}'\gamma_{\mathrm{L}}'' + \left(\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\right)^2 \gamma_{\mathrm{L}}'\gamma_{\mathrm{L}}'''\right] \tag{A.4}$$

$$\begin{aligned}\chi''''(\gamma_{\mathrm{L}}) = \left(\frac{\omega}{\bar{\alpha}_{\mathrm{S}}}\right)^5 \frac{1}{\gamma_{\mathrm{L}}'^7} \Bigg[&24\gamma_{\mathrm{L}}'^3 + 36\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\gamma_{\mathrm{L}}'^2\gamma_{\mathrm{L}}'' + 30\left(\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\right)^2 \gamma_{\mathrm{L}}'\gamma_{\mathrm{L}}''^2 - 8\left(\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\right)^2 \gamma_{\mathrm{L}}'^2\gamma_{\mathrm{L}}''' + \\ &+ 15\left(\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\right)^3 \gamma_{\mathrm{L}}''^3 - 10\left(\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\right)^3 \gamma_{\mathrm{L}}'\gamma_{\mathrm{L}}''\gamma_{\mathrm{L}}''' + \left(\frac{\bar{\alpha}_{\mathrm{S}}}{\omega}\right)^3 \gamma_{\mathrm{L}}'^2\gamma_{\mathrm{L}}''''\Bigg].\end{aligned} \tag{A.5}$$

Conversely, the derivatives of γ_{L} are given in terms of $\chi(\gamma_{\mathrm{L}})$ by

$$\gamma_{\mathrm{L}}' = -\frac{\chi^2}{\chi'} \tag{A.6}$$

$$\gamma_{\mathrm{L}}'' = \frac{\chi^4}{\chi'^3}\left(2\chi\chi'^2 - \chi''\right) \tag{A.7}$$

$$\gamma_{\mathrm{L}}''' = \frac{\chi^6}{\chi'^5}\left[-6\chi^2\chi'^4 + 6\chi\chi'^2\chi'' - 3\chi''^2 + \chi'\chi'''\right] \tag{A.8}$$

$$\begin{aligned}\gamma_{\mathrm{L}}'''' = \frac{\chi^8}{\chi'^7}\Big[&24\chi^3\chi'^6 - 15\chi''^3 + 36\chi\chi'^2\chi''^2 - 36\chi^2\chi'^4\chi'' - 12\chi\chi'^3\chi''' \\ &+10\chi'\chi''\chi''' - \chi'^2\chi''''\Big].\end{aligned} \tag{A.9}$$

References

[1] R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and collider physics* Cambridge University Press, Cambridge 1996.

[2] V.S. Fadin, E.A. Kuraev, L.N. Lipatov, *Phys. Lett.* **B 60** (1975) 50;
Ya.Ya. Balitsky, L.N. Lipatov, *Sov. J. Nucl. Phys.* **28** (1978) 822;
L.N. Lipatov, in *Perturbative QCD*, A.H. Müller ed. World Scientific, 1989.

[3] J.R. Forshaw, D.A. Ross, *Quantum chromodynamics and the pomeron* Cambridge University Press, Cambridge 1997.

[4] J.R. Forshaw and A. Sabio Vera, *Phys. Lett.* **B 440** (1998) 141 [hep-ph/9806394].

[5] B.R. Webber, *Phys. Lett.* **B 444** (1998) 81 [hep-ph/9810286].

[6] C. Ewerz and B.R. Webber, *J. High Energy Phys.* **04** (1999) 022 [hep-ph/9904244].

[7] M. Ciafaloni, *Nucl. Phys.* **B 296** (1988) 49.

[8] S. Catani, F. Fiorani, G. Marchesini, *Phys. Lett.* **B 234** (1990) 339, *Nucl. Phys.* **B 336** (1990) 18; S. Catani, F. Fiorani, G. Marchesini, G. Oriani, *Nucl. Phys.* **B 361** (1991) 645.

[9] G.P. Salam, *J. High Energy Phys.* **03** (1999) 009 [hep-ph/9902324].

[10] M. Ciafaloni, D. Colferai, G. Salam, *Renormalization group improved small-x equation*, hep-ph/9905566.

[11] V.S. Fadin, L.N. Lipatov, *Phys. Lett.* **B 429** (1998) 127 [hep-ph/9802290] and references therein.

[12] M. Ciafaloni, G. Camici, *Phys. Lett.* **B 430** (1998) 349 [hep-ph/9803389] and references therein.

[13] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, *Eur. Phys. J.* **C 4** (1998) 463 [hep-ph/9803445].

[14] J. Kwiecinski, C.A.M. Lewis and A.D. Martin, *Phys. Rev.* **D 54** (1996) 6664 [hep-ph/9606375].

[15] C.R. Schmidt, *Phys. Rev. Lett.* **78** (1997) 4531 [hep-ph/9612454].

[16] L.H. Orr and W.J. Stirling, *Phys. Rev.* **D 56** (1997) 5875 [hep-ph/9706529].